Coarse geometry of homeomorphism groups: Classifying countable Stone spaces

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Abstract

Towards developing the tools of geometric group theory for non-locally compact topological groups, we give one of the first complete classifications of a family of such groups up to coarse equivalence, and when possible, up to quasi-isometry. In a previous paper, we placed the homeomorphism groups of countable Stone spaces into three classes: coarsely bounded, unbounded yet generated by a coarsely bounded set, and unbounded but not generated by any coarsely bounded set. Now we show that these are the coarse equivalence classes: any two groups within one of these classes are in fact coarsely equivalent.

Furthermore, we show that groups in the second class are quasi-isometric to the *Hamming cube,* the space comprising infinite binary sequences with finitely many nonzero entries equipped with the Hamming distance. As part of the proof, we show that infinite Hamming graphs over finite alphabets are all bi-Lipschitz equivalent.

1 Introduction

Countable Stone spaces are classified up to homeomorphism by two pieces of data: the Cantor-Bendixson rank of the space, which is a countable ordinal α, and an integer $n \geq 1$, which is the number of points of maximal rank. We write $X_{\alpha,n}$ to denote the corresponding Stone space and study its group of symmetries, $\mathrm{Homeo}(X_{\alpha,n})$, which is a topological group equipped with the compact-open topology. In [BDHL25, Theorem A], we showed that these groups all have metrizable coarse structures that fall into three categories: coarsely bounded, unbounded and generated by a coarsely bounded subset, or unbounded but not generated by any coarsely bounded subset. In this paper we show that these are also the coarse equivalence classes.

Theorem A. Let $X_{\alpha,n}$ be a countable Stone space with $\alpha > 0$. The groups $\mathrm{Homeo}(X_{\alpha,n})$ equipped with the compact-open topology fall into three coarse equivalence classes, which are determined by α and n as follows.

1. If $n = 1$, then for all α, $\mathrm{Homeo}(X_{\alpha,1})$ is coarsely bounded and hence quasi-isometric to a point.
2. If $n > 1$ and α is a successor ordinal, then $\mathrm{Homeo}(X_{\alpha,n})$ is quasi-isometric to the countably infinite Hamming cube.
3. If $n > 1$ and α is a limit ordinal, then $\mathrm{Homeo}(X_{\alpha,n})$ is coarsely equivalent to a level set of vertices of a 1-ended tree of Hamming cubes (see Definition 5.4).

In particular, all unbounded groups that are generated by a coarsely bounded subset fall into the second category and are quasi-isometric to each other. This gives perhaps the first complete quasi-isometric classifications for an infinite family of non-locally compact topological groups.

Moreover, our result also supplies explicit geometric models. The *countably infinite Hamming cube* is the graph whose vertices correspond to binary sequences with finitely many 1's. Two vertices are connected by an edge if they differ in a single coordinate. Geometrically, this is the 1-skeleton of a connected component of an infinite dimensional cube. It includes all of the finite Hamming cubes as subgraphs. The *one-ended tree of Hamming cubes* arranges countably many countably infinite Hamming cubes in the pattern of a 1-ended tree with countably many leaves where each non-leaf vertex has countably infinite valence. See Section 5.2 for the formal definition of this space.

Prior to this work, explicit descriptions of coarse structures associated to non-locally compact topological groups were only known in the case of automorphism groups of locally infinite trees and the isometry group of Urysohn space [Ros22, Section 6.5], homeomorphisms groups of compact manifolds [MR18], and the mapping class group of the plane minus a Cantor set [Cal09, Bav16, MR23, SC24].

1.1 A guiding example

The Stone space $X_{1,2}$ is homeomorphic to $\hat{\mathbb{Z}} = \mathbb{Z} \cup \{\pm\infty\}$, the 2-point compactification of $\mathbb{Z}$. The two maximal points are $\pm\infty$, while all other points are isolated in the topology. To illustrate the main ideas of this paper, we describe a graph Γ that captures the quasi-isometry type of $\mathrm{Homeo}(\hat{\mathbb{Z}})$ and is isomorphic to the Hamming cube. A general proof is done rigorously in Section 4.1.

The vertices of Γ correspond to partitions of $\hat{\mathbb{Z}}$ into two clopen sets separating $-\infty$ and $+\infty$. Two vertices are connected by an edge if one partition can be constructed from the other by moving a singleton. The prototypical example of adjacent vertices $\mathcal{P}$ and $\mathcal{R}$ is given by

$$P_- = \{x \leq 0\} \quad \text{and} \quad P_+ = \{x > 0\} \qquad \text{while} \qquad R_- = \{x < 0\} \quad \text{and} \quad R_+ = \{x \geq 0\}.$$

Here $P_- = R_- \sqcup \{0\}$ and $P_+ \sqcup \{0\} = R_+$. These are basic examples of what we call *dividing partitions* and what it means for two partitions to *differ by a shift*. See Figure 1 for a an example of four vertices in Γ and the corresponding edges between them.

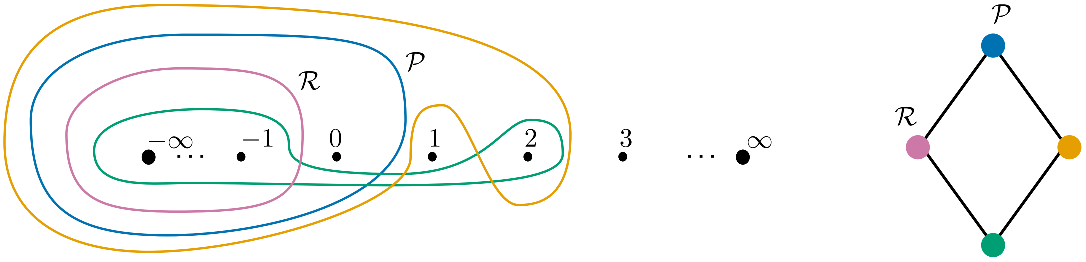

Figure 1: A small piece of the graph Γ. The partitions $\mathcal{P}$ and $\mathcal{R}$ defined above are labeled in blue and pink, respectively. Partitions are represented as Jordan curves in the plane so that the interior of the curve contains $-\infty$.

Because homeomorphisms fix the set of maximal points, the group $\mathrm{Homeo}(\hat{\mathbb{Z}})$ acts on Γ; in fact it acts continuously. A homeomorphism f acts by sending a partition $\mathcal{P} = P_- \sqcup P_+$ to the induced partition

$$f.\mathcal{P} = \{f(x) : x \in P_-\} \sqcup \{f(x) : x \in P_+\}.$$

In fact, in [BDHL25] we showed that Γ is a *Cayley-Abels-Rosendal graph* for $\mathrm{Homeo}(\hat{\mathbb{Z}})$, meaning that the group has a canonical quasi-isometry type which is captured by the graph.

Now we aim to see that Γ is isometric to a Hamming graph. Fix a basepoint partition, $\mathcal{P}_0$, say the partition with $P_- = \{x \leq 0\}$ and $P_+ = \{x > 0\}$ from above. Let $\mathcal{Q} = Q_- \sqcup Q_+$ be any other partition. Because the partitions are clopen sets, there is an $N > 0$ such that

$$Q_- \cap [-\infty, -N] = P_- \cap [-\infty, -N] \quad \text{and} \quad Q_+ \cap [N, \infty] = P_+ \cap [N, \infty].$$

In particular, $\mathcal{P}_0$ and $\mathcal{Q}$ differ on finitely many points. Moving these points one at a time shows that the number of such points is exactly their distance in Γ. In fact, recording the points that differ between $\mathcal{P}_0$ and $\mathcal{Q}$ allows us to label the vertices of Γ by finite subsets of $\mathbb{Z}$. Two vertices are adjacent when their corresponding subsets differ by a single element. Thus for example, the partition $\mathcal{R}$ given above corresponds to the subset $\{0\}$. This viewpoint shows us that Γ is naturally isomorphic to the poset graph of finite subsets of $\mathbb{Z}$. See Figure 2 for an illustration of this.

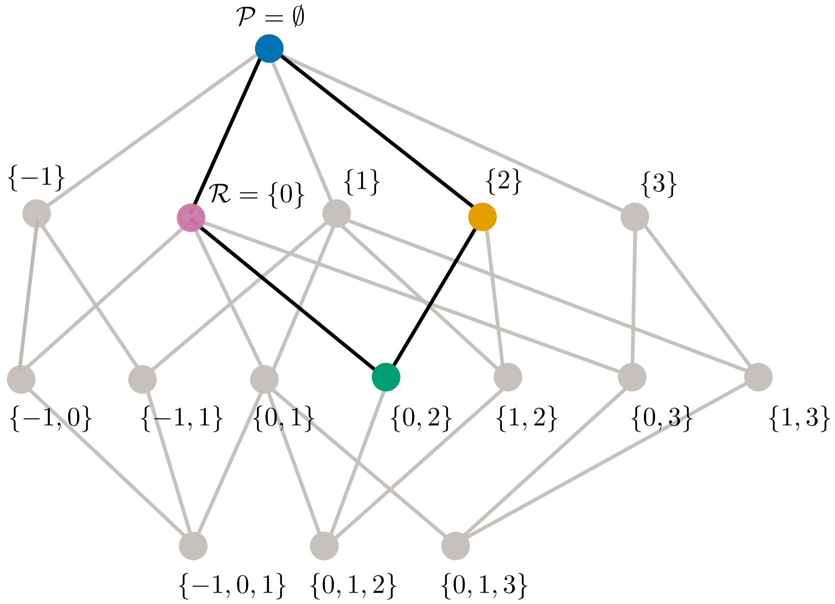

Figure 2: A piece of the graph Γ realized as the poset graph of finite subsets of $\mathbb{Z}$. The square piece highlighted in color is the square of Γ pictured in Figure 1.

The characteristic function of a subset $S \subset \mathbb{Z}$ allows us to think of vertices of the poset graph as $\mathbb{Z}$-indexed sequences in $\{0, 1\}$. In this perspective, an edge of Γ corresponds to two sequences differing in one coordinate; thus Γ is a Hamming graph. See Figure 3 for a realization of a piece of Γ as this Hamming graph. Finally, fixing any bijection $\mathbb{Z} \to \omega$ allows one to reindex $\bigoplus_{n \in \omega} \mathbb{Z}/2\mathbb{Z}$ and realize it as the standard countably infinite Hamming cube.

1.2 History and motivation

Much of modern geometric group theory has developed around Gromov's [Gro87, Gro93] program of studying a group via its *coarse geometric* properties. Traditionally this has focused on finitely generated (discrete) groups. This is because finitely generated groups admit well-defined quasi-isometry classes of left-invariant metrics, namely those given by the word metric for any (and hence all) finite generating sets. When moving to *topological* groups, which need not be finitely generated, one now can ask for a canonical class of left-invariant metrics that additionally generate the group topology, or at least are continuous with respect to it. Gromov's program was first extended to the class of locally compact topological groups (see e.g. [CdlH16]). Here *compact* generating sets and their associated word metrics yield the canonical quasi-isometry type, and

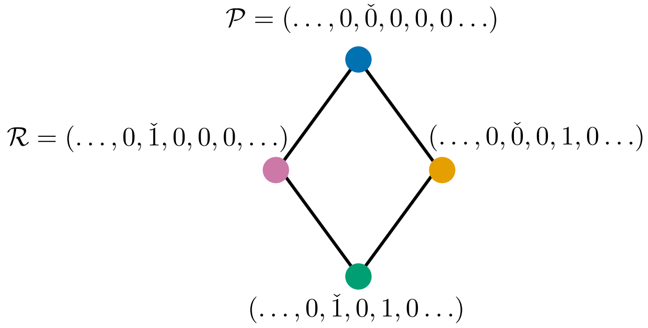

Figure 3: A piece of the graph Γ realized as the countably infinite Hamming cube. This shows the same square of Γ pictured in Figure 1. The check marks indicate the 0th coordinate.

although the word metric is always discrete, it is quasi-isometric to a metric that is compatible with the group topology.

More recently, Rosendal [Ros22], building on work of Roe [Roe03], has extended this framework to general topological groups. The substitute for finiteness or compactness here is the property of being *coarsely bounded*. A subset of a topological group is *coarsely bounded*, or CB, if it has finite diameter in any continuous (pseudo-)metric on the group. If a Polish group has a CB neighborhood of the identity, i.e. if it is *locally CB*, then it has a metrizable coarse structure. Additionally, if a Polish topological group admits a CB generating set, the word metric associated to this generating set (and hence any analytic CB generating set) gives a well-defined quasi-isometry class of metrics on the group. Furthermore, this class has a representative that generates the given topology on the group. See also [BDHL25, Sections 1 & 2] for a survey on this perspective.

In our previous paper [BDHL25], we introduced *Cayley–Abels–Rosendal graphs* for topological groups. Similar to the Cayley graph of a discrete group with respect to a finite generating set, Cayley–Abels–Rosendal graphs provide a concrete geometric model for the quasi-isometry type of a topological group that admits one. Geometric group theorists have long used Cayley graphs to great effect, and this paper continues in that tradition.

When a group G has metrizable coarse structure but is not generated by a coarsely bounded set, no Cayley–Abels–Rosendal graph exists, but it is still desirable to have a geometric model for the coarse structure on the group. This was done for non-Archimedean Polish groups by Kopreski–Shaji [KS25], who introduced *coarse Cayley–Abels–Rosendal* graphs, which are metric graphs with a discrete set of edge lengths. The vertex set of a coarse Cayley–Abels–Rosendal graph Γ for a group G, when equipped with the subspace metric, is coarsely equivalent to G.

In Section 5.1, building on Kopreski–Shaji's work and using Bass–Serre theory, we construct a countable graph that allows us to compute the coarse structure of G. This graph will have infinitely many orbits of vertices, which are naturally organized by an exhaustive, countably infinite chain of proper, open subgroups. Nonetheless, the orbit of any vertex still captures the coarse structure of G.

1.2.1 The discrete topology

The statement of Theorem A specifies that $\mathrm{Homeo}(X_{\alpha,n})$ is equipped with the compact-open topology. For a topological group in general, when we write "quasi-isometric," this should always be taken in reference to a maximal *continuous* left-invariant (pseudo-)metric. Quasi-isometry is thus a *topological group* invariant rather than an *abstract* group invariant. Asking for the latter is equivalent to considering coarse structures, á la [Ros22], on groups equipped with the *discrete* topology. The notion of coarse boundedness (CB) in a discrete group is also referred to as *strong* *boundedness* (SB) and this framework has been expanded on in [Vla26]. In particular, an SB-generated group also admits a well-defined quasi-isometry class of metrics. In [Vla26], building on work in [BCM$^+$24], Vlamis shows that when α is a successor ordinal, Homeo($X_{\alpha,n}$) is SB-generated. In fact, Vlamis's work shows that the coarsely bounded generating sets given in [BDHL25] are strongly bounded. Thus, combining Theorem A with [Vla26, Theorem 6.1] we can obtain the following stronger result.

Corollary B. If $X_{\alpha,n}$ is a countable Stone space with $\alpha > 0$ a successor ordinal and $n > 1$, then Homeo($X_{\alpha,n}$) equipped with the discrete topology is quasi-isometric to the countable infinite Hamming cube.

1.2.2 Connections to surface mapping class groups

One motivation for this project comes from attempts at obtaining a quasi-isometry classification for mapping class groups of infinite-type surfaces; see for example Problem 2.46(b) of the new Kirby problem list [BKR26]. When equipped with the (quotient of) the compact–open topology, these groups are no longer finitely generated and discrete, but instead are non-locally compact topological groups. By the work of Mann–Rafi [MR23], it is known that many such mapping class groups are CB-generated and hence admit well-defined quasi-isometry classes of metrics. However, little is known about the quasi-isometry classification of these groups. So far, one can distinguish those mapping class groups that are themselves CB from those that are not [MR23, JM23] and some examples that are hyperbolic groups [SC24]. Some quasi-isometry invariants such as asymptotic dimension, coarse rank, and divergence bounds are also known [GRV21, KS25, BNQR26]. A full classification, however, is nowhere near in sight.

The end spaces of surfaces are second countable Stone spaces. In fact, every such Stone space is the end space of an infinite-type surface [Ker23, Ric63]. Mapping class groups act on their end spaces continuously by homeomorphisms. Therefore, given an infinite-type surface S with end space $E(S)$ and end space accumulated by genus $E_g(S)$, we have the following short exact sequence of topological groups.

$$1 \longrightarrow \mathrm{PMCG}(S) \longrightarrow \mathrm{MCG}(S) \longrightarrow \mathrm{Homeo}(E(S), E_g(S)) \longrightarrow 1$$

where MCG(S) denotes the mapping class group, PMCG(S) denotes the closed subgroup of "pure" mapping classes, and Homeo($E(S), E_g(S)$) denotes the closed subgroup of Homeo($E(S)$) that preserves the closed subset $E_g(S) \subset E(S)$. This suggests that a natural starting point for understanding the coarse geometric structure of MCG(S) is to first understand the coarse geometric structure of Homeo(X) for X a second countable Stone space.

Our classification does not imply an analogous quasi-isometric classification for mapping class groups. In Section 6, we see that the quotient map given above is *not* a quasi-isometry (Lemma 6.1). Unlike in the group Homeo($X_{\alpha,n}$), we also show that the Cantor-Bendixson derivative on end spaces does *not* induce a quasi-isometry between mapping class groups (Lemma 6.2), even when $E_g(S) = \varnothing$. These results, together with Theorem A, suggest that one *must* make use of the topology/geometry of the surface when classifying mapping class groups up to quasi-isometry.

The results in Section 6 do not actually cover all infinite-type surfaces. Curiously, the arguments fail for so-called "translatable" or "avenue" surfaces (á la [SC24] and [HQR22], respectively). One such example of these surfaces is $\Sigma_{1,2}$, the zero genus surface with end space homeomorphic to $X_{1,2}$. This raises the following question. Here $\zeta(\Sigma)$ is a measure of complexity as defined in [BNQR26].

Question 1. Let Σ be a stable surface with $\zeta(\Sigma) = 2$ whose mapping class group is CB-generated but not coarsely bounded. Is PMCG(Σ) coarsely bounded in MCG(Σ)?

More specifically, let $\Sigma_{1,2}$ be the genus zero surface with end space $X_{1,2}$. Is PMCG($\Sigma_{1,2}$) coarsely bounded inside of MCG($\Sigma_{1,2}$)?

1.2.3 Connections to graph mapping class groups

Our results more directly extend to the setting of mapping class groups of locally finite, infinite graphs. We refer the reader to [AKB25] for the relevent definitions. The mapping class group of a tree is isomorphic (as a topological group) to the homeomorphism group of its end space, i.e. a second countable Stone space. In this sense, these mapping class groups can be seen as a direct extension of the groups studied here. Additionally, [DHK23, DHK25] show that the (pure) mapping class groups of many graphs do admit a well-defined quasi-isometry class of metrics. Provided that one first verifies that the same is true for the full mapping class groups, a natural question is the following.

Question 2. Let Γ be a locally finite, infinite graph with isolated ends not accumulated by loops. Let Γ' be the graph obtained from Γ by forgetting all such isolated ends. Does this forgetful map induce a quasi-isometry between $\mathrm{MCG}(\Gamma)$ and $\mathrm{MCG}(\Gamma')$?

This forgetful map is an analogue of the Cantor-Bendixson derivative and by Corollary 3.4 we know that the answer to this question is "yes" when Γ is a tree with countable end space. However, by Lemma 6.2, we know that the answer to the analogous question in the surface case is "no." These mapping class groups of graphs often exhibit behavior somewhere between homeomorphism groups of Stone spaces and mapping class groups of surfaces.

1.3 Organization and outline

Here we provide both an outline of the paper and the proof of Theorem A. First, in Section 2 we recall background notions of coarse geometry and aspects of our work in [BDHL25] necessary for this paper.

In Section 3, mostly using ideas from [BDHL25], we show that the Cantor-Bendixson derivative on a countable Stone space induces a coarse equivalence of homeomorphism groups. For successor ordinals, this immediately allows one to reduce Theorem A to the case where $\alpha = 1$. We tackle the rest of the successor ordinal case in Section 4. Here we first exhibit isomorphisms between the Cayley-Abels-Rosendal graphs built in [BDHL25] with infinite Hamming graphs over finite alphabets (Section 4.1). Then we study the geometry of these Hamming graphs and show that they are all bi-Lipschitz equivalent (Section 4.2), thus finishing the proof of Theorem A for successor ordinals.

Next we turn to limit ordinals in Section 5. Since the corresponding homeomorphism groups are no longer CB-generated, they do not admit Cayley-Abels-Rosendal graphs. We first construct "trees of Cayley-Abels-Rosendal graphs" that model the coarse structure of our groups. This construction is similar to the construction of coarse Cayley-Abels-Rosendal graphs of [KS25] and we make use of a number of their tools. In Section 5.1 we provide a general construction of these graphs for any non-Archimedean Polish group that is locally CB but not CB-generated.

When λ is a limit ordinal, there is a family of proper open subgroups $G_n \leq \mathrm{Homeo}(X_{\lambda,n})$ that exhaust $\mathrm{Homeo}(X_{\lambda,n})$. Bass–Serre theory provides an action of $\mathrm{Homeo}(X_{\lambda,n})$ on a 1-ended tree with all stabilizers finitely generated over a fixed coarsely bounded identity neighborhood. We show in Proposition 5.6 that after blowing up these vertices with copies of infinite Hamming graphs, we produce a new "tree of Hamming graphs" on which $\mathrm{Homeo}(X_{\lambda,n})$ acts in such a way that any orbit map is a coarse equivalence, following [KS25, Ros22].

In Section 5.2 we formally define these "trees of Hamming graphs" as the appropriate analog of infinite Hamming graphs and again show that our constructions are all bi-Lipschitz equivalent. Finally in Section 5.3 we combine these two pieces to see that for any limit ordinal, the corresponding homeomorphism group is coarsely equivalent to a tree of Hamming graphs and hence all such groups are coarsely equivalent.

If $n = 1$, we showed in [BDHL25, Corollary 14] that the group Homeo$(X_{\alpha,1})$ is coarsely bounded. Thus, groups in this class are all coarsely equivalent to a point, and we have finished the proof of Theorem A. To conclude we discuss connections to surfaces in Section 6.

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The statements and the outline of proofs in Section 4.2 were obtained by prompting an LLM, ChatGPT 5.5-Pro, and suggested to us by Christian Rosendal. The proofs in Section 4.2 were written by the authors upon reading the output provided by this prompting and the authors take full responsibility for the content of the paper.

Contents

2 Background

This section contains background material necessary to what follows. There is a brief review of the machinery of Roe [Roe03] and Rosendal [Ros22] for coarse geometry on topological groups. We discuss ordinals only to set notation, recall the spaces $X_{\alpha,n}$ and the graphical models we constructed in [BDHL25].

2.1 Coarse geometry

We direct the reader to Rosendal's book [Ros22] for more thorough background on coarse geometry for general topological groups (see also [BDHL25, Sections 1 & 2], [Lym25] or [KS25] for an abridged overview). We remind the reader only of the basics, partially to set notation.

Every topological group G admits a canonical coarse structure (á la [Roe03]), the *left-coarse structure* by considering all possible continuous left-invariant (pseudo)-metrics on G; equivalently all continuous, isometric actions of G on metric spaces. A subset $A \subset G$ is *coarsely bounded*, or CB, if it is bounded in this coarse structure. This means that A has bounded diameter in any continuous left-invariant pseudometric on G. See [Ros22, Proposition 2.15] for other characterizations.

Adding the assumption that G is Polish (meaning separable and completely metrizable), the coarse structure on G is metrizable if and only if G has a coarsely bounded neighborhood of the identity, i.e. G is *locally CB* [Ros22, Theorem 2.38]. If G is locally CB then, by separability of G, it is also countably generated over a CB neighborhood of the identity. That is, there exists a CB neighborhood of the identity H in G and a countable set S in G so that H and S algebraically generate G. Furthermore, G admits a well-defined quasi-isometry class of metrics if and only if G is generated by a coarsely bounded set [Ros22, Proposition 2.72 and Theorem 2.73], which in the case of Polish groups may be taken to be a CB neighborhood of the identity H together with a *finite* set S.

A map $f\colon (X, d_X) \to (Y, d_Y)$ between metric spaces is a *quasi-isometry* when there exist positive constants (L, K, C) such that for all x and y in X, we have

$$\frac{1}{L} d_X(x, y) - K \leq d_Y(f(x), f(y)) \leq L d_X(x, y) + K$$

and for each $y \in Y$ there exists $x \in X$ such that $d_Y(f(x), y) \leq C$.

The latter condition above says that the map f is *cobounded.* Replacing the affine upper control $t \mapsto Lt + K$ in the inequality $d_Y(f(x), f(y)) \leq L d_X(x, y) + K$ with an arbitrary increasing function that limits to infinity, $\eta\colon \mathbb{R} \to \mathbb{R}$ of $d_X(x, y)$, one obtains the definition of a *bornologous* map of metric spaces. Doing the same for the inequality $\frac{1}{L} d_X(x, y) - K \leq d_Y(f(x), f(y))$, one defines an *expanding* map.

A *coarse equivalence* of metric spaces is a map that is bornologous, expanding and cobounded. Since our topological groups have metrizable coarse structure, these definitions suffice to characterize coarse equivalence. When a Polish group is CB generated, coarse equivalence turns out to recover the notion of quasi-isometry. Indeed, by [Ros22, Theorem 2.40] and [Roe03, Proposition 2.57], when a Polish group is CB generated, its coarse structure can be realized by a quasi-geodesic metric space. By [Ros22, Lemma 2.66] a coarse equivalence between quasi-geodesic metric spaces is in fact a quasi-isometry.

Here is one final proposition that we will make use of repeatedly.

Proposition 2.1. *[Ros22, Proposition 4.37] Let K be a closed normal subgroup of a Polish group G. The quotient map $G \to G/K$ is a coarse equivalence if and only if K is coarsely bounded in G. If G is generated by a CB set, then the quotient map is a quasi-isometry.*

A *Cayley–Abels–Rosendal* graph for a Polish group G is a countable, connected graph Γ on which G acts continuously with coarsely bounded stabilizers, one orbit of vertices and finitely many orbits of edges. We introduced Cayley–Abels–Rosendal graphs in our previous paper [BDHL25]. When G admits a Cayley–Abels–Rosendal graph, G is generated by an open, coarsely bounded set. The group G and the graph Γ are quasi-isometric.

2.2 Ordinals

An *ordinal* is the order isomorphism class of a well-ordered set. The class of ordinals is *itself* well-ordered. Therefore, if α and β are ordinals, then we write $\beta < \alpha$ to say that β occurs before

α in the associated well-order.

An ordinal α is a *successor ordinal* if there is an ordinal β with $\alpha = \beta + 1$. That is, $\beta < \alpha$, and α is the smallest ordinal with this property; its existence is guaranteed by the well ordering of ordinals.

A nonzero ordinal that is not a successor ordinal is called a *limit ordinal.* We denote the first infinite ordinal by $\omega = \{0, 1, 2, \ldots\}$; it is also the first limit ordinal. We will not consider 0 to be either a limit or successor ordinal.

In this paper we only use countable ordinals. If λ is a countable limit ordinal, then we can choose an increasing sequence $\alpha_0 < \alpha_1 < \cdots < \lambda$ for which $\lim_{k\to\infty} \alpha_k = \lambda$. Such a sequence is called a *cofinal* sequence.

2.3 The spaces $X_{\alpha,n}$

Definition 2.2. A *Stone space* is a space that is compact, Hausdorff and totally disconnected.

Stone spaces X are either perfect or contain an isolated point. We are interested in Stone spaces X with countably many points; the only perfect, countable Stone space is empty.

Definition 2.3. The *Cantor–Bendixson derivative* of a space X, denoted X', is the closed subset of X remaining after all isolated points of X are removed. More generally, the operation of Cantor–Bendixson derivative can be iterated by transfinite induction:

Base case: $X^{(0)} = X$

Successor case: $X^{\alpha+1} = (X^{\alpha})'$

Limit case: If $\lambda = \lim\limits_{k\to\infty} \alpha_k$, then $X^{(\lambda)} = \bigcap_k X^{(\alpha_k)}$.

If X is a countable Stone space, then for some countable ordinal α, the space $X^{(\alpha)}$ is empty. By the finite intersection property, the smallest such ordinal is a successor ordinal $\alpha+1$, and the space $X^{(\alpha)}$ is nonempty but has finitely many points, say n. These two pieces of data: the ordinal α and the number $n \geq 1$ characterize X up to homeomorphism [MS20]. We denote countable Stone spaces $X_{\alpha,n}$. We call the points of $X^{(\alpha)}_{\alpha,n}$ the *maximal points* of $X_{\alpha,n}$. The terminology originates from the Mann-Rafi preorder on ends of an infinite type surface defined in [MR23, Section 4].

A concrete model for $X_{\alpha,n}$ can be given by the ordinal $\omega^{\alpha} \cdot n + 1$ equipped with the order topology. However, we will not make use of this order structure, and only consider $X_{\alpha,n}$ as a topological space.

2.4 Graphical models for $\mathrm{Homeo}(X_{\alpha,n})$

In this subsection we summarize necessary constructions and results from our first paper, [BDHL25].

Lemma 2.4 ([BDHL25] Corollary 14)**.** *The group* $\mathrm{Homeo}(X_{\alpha,1})$ *is coarsely bounded in itself, whence it is coarsely bounded in any topological group into which it embeds continuously.*

Definition 2.5. A *dividing partition* of $X_{\alpha,n}$ is a clopen partition

$$\mathcal{P} = P_1 \sqcup \cdots \sqcup P_n$$

into n sets, such that each P_i contains a single maximal point. As such, in a dividing partition $\mathcal{P}$, each partition element P_i is homeomorphic to $X_{\alpha,1}$. In our previous paper, these were called "good partitions" and were defined only for successor ordinals.

The stabilizers of clopen partitions of $\mathrm{Homeo}(X_{\alpha,n})$ form a neighborhood basis of the identity [BL24].

Lemma 2.6. *The stabilizer in* $\mathrm{Homeo}(X_{\alpha,n})$ *of a dividing partition has an open, finite-index subgroup isomorphic to* $\prod_{i=1}^{n}\mathrm{Homeo}(X_{\alpha,1})$*. As such, it is an open, coarsely bounded subgroup.*

Proof. This stabilizer is precisely the subgroup used in [BDHL25, Corollary 15] to show that the group $\mathrm{Homeo}(X_{\alpha,n})$ is locally bounded. ☐

Definition 2.7. Fix $X = X_{\alpha,n}$. If $\beta < \alpha$ is a countable ordinal and $n > 1$, we say that two dividing partitions $\mathcal{P} = P_1 \sqcup \cdots \sqcup P_n$ and $\mathcal{Q} = Q_1 \sqcup \cdots \sqcup Q_n$ of X *differ by a β-shift* if there is a clopen subspace S homeomorphic to $X_{\beta,1}$ such that, after possibly swapping the roles of $\mathcal{P}$ and $\mathcal{Q}$ and reindexing, we have

$$P_i = Q_i \text{ for } 1 < i < n, \quad P_1 = Q_1 \sqcup S \quad \text{and} \quad P_n \sqcup S = Q_n.$$

In other words, the subspace S shifts “out of” P_1 and “into” P_n to create the new partition $\mathcal{Q}$. When $\alpha = \beta + 1$ is a successor ordinal, we call a β-shift a *maximal shift.* We use the language of shifting because in [BDHL25] we show that there is always an infinite order homeomorphism of $X_{\alpha,n}$ that takes $\mathcal{P}$ to $\mathcal{Q}$.

Definition 2.8. Fix an ordinal α and integer $n > 1$. Given a countable ordinal $\beta < \alpha$, define the graph $\Gamma(\alpha, n)$ to be the simplicial graph whose vertex set is the collection of dividing partitions of $X_{\alpha,n}$, where two partitions are connected by an edge when they differ by a β-shift.

In [BDHL25, Section 4.3] we studied the case when $\alpha = \beta + 1$ is a successor ordinal. We showed that the graph $\Gamma(\alpha, n)$ whose vertices are dividing partitions and where edges correspond to maximal shifts, is a Cayley–Abels–Rosendal graph for $\mathrm{Homeo}(X_{\alpha,n})$.

3 The Cantor–Bendixson derivative

The following lemma is implicit in the proof of [BDHL25, Lemma 17].

Lemma 3.1. *Suppose* $\mathcal{P} = P_1 \sqcup \cdots \sqcup P_n$ *and* $\mathcal{Q} = Q_1 \sqcup \cdots \sqcup Q_n$ *are dividing partitions of* $X_{\alpha,n}$*. Fix* $\beta < \alpha$*. If for all* i*, each rank-*β *point of* P_i *belongs to* Q_i *and vice versa, then* $\mathcal{P}$ *and* $\mathcal{Q}$ *differ by at most* $n(n-1)$ β*-shifts.*

Proof. We will construct a path of β-shifts from $\mathcal{P}$ to $\mathcal{Q}$ of length at most $n(n-1)$.

Choose a pair of distinct indices (i, j) in $\{1, \ldots, n\}$. We will construct a new partition

$$\mathcal{R} = R_1 \sqcup \cdots \sqcup R_n$$

such that

1. $R_k = P_k$ if $k \neq i, j$,
2. $R_i \cap (P_i \sqcup P_j) \cap (Q_i \sqcup Q_j) = Q_i \cap (P_i \sqcup P_j)$,
3. and therefore $R_j \cap (P_i \sqcup P_j) \cap (Q_i \sqcup Q_j) = Q_j \cap (P_i \sqcup P_j)$.

In other words, points of P_j which move to Q_i and points of P_i which move to Q_j will belong to R_i and R_j, respectively, while points common to P_i and Q_i remain in R_i and similarly for R_j.

The partition $\mathcal{R}$ will by construction differ from $\mathcal{P}$ by at most two β-shifts. Repeating this construction for the pair $\mathcal{R}, \mathcal{Q}$ and a different unordered pair of indices will produce $\mathcal{Q}$ after at most $\frac{n(n-1)}{2}$ repetitions, whence the claim.

Here is the construction: choose arbitrarily a rank-β point x in $P_i \cap Q_i$ and a clopen neighborhood Y of x in $P_i \cap Q_i$ homeomorphic to $X_{\beta,1}$. By the classification of countable Stone spaces, because both $P_i \cap Q_j$ and $P_j \cap Q_i$ contain only points of rank strictly below β, both $Y_1 = Y \sqcup (Q_j \cap P_i)$ and $Y_2 = Y \sqcup (Q_i \cap P_j)$ are homeomorphic to $X_{\beta,1}$. The partition $\mathcal{S} = S_1 \sqcup \cdots \sqcup S_n$ with

$$S_k = P_k \text{ if } k \neq i, j, \quad S_i = P_i \setminus Y_1, \quad \text{and} \quad S_j = P_j \sqcup Y_1$$

therefore differs from $\mathcal{P}$ by a β-shift. Similarly the partition $\mathcal{R}$ with

$$R_k = P_k \text{ if } k \neq i, j, \quad R_i = S_i \sqcup Y_2, \quad \text{and } R_j = S_j \setminus Y_2$$

differs from $\mathcal{S}$ be a β-shift and has the desired properties above. This completes the claim and with it the proof. ☐

The lemma has the following corollary.

Corollary 3.2. *Any subset* $A \subset \mathrm{Homeo}(X_{\alpha,n})$ *that acts trivially on the set of rank-*β *points of* $X_{\alpha,n}$ *for* $\beta < \alpha$ *is coarsely bounded.*

Proof. By Lemma 3.1, if $\mathcal{P}$ is a dividing partition and $f \in \mathrm{Homeo}(X_{\alpha,n})$ is a β-shift on $\mathcal{P}$, the set A is contained in the coarsely bounded set $(\{f^{\pm 1}\}H)^{n(n-1)}$. ☐

Now we have two more immediate corollaries that will be utilized in the successor ordinal case.

Corollary 3.3. *The kernel of the action of* $\mathrm{Homeo}(X_{\beta+1,n})$ *on the closed subset* $X^{(\beta)}_{\beta+1,n} \cong X_{1,n}$ *is coarsely bounded in* $\mathrm{Homeo}(X_{\beta+1,n})$*.*

Proof. An element f is in the kernel K of the action of $\mathrm{Homeo}(X_{\beta+1,n})$ on $X^{(\beta)}_{\beta+1,n}$ if and only if it fixes every point of rank β. ☐

Corollary 3.4. *For any countable ordinal* β*, the group* $\mathrm{Homeo}(X_{\beta+1,n})$ *is quasi-isometric to* $\mathrm{Homeo}(X_{1,n})$*.*

Proof. The action of $\mathrm{Homeo}(X_{\beta+1,n})$ on the closed subset $X^{(\beta)}_{\beta+1,n} \cong X_{1,n}$ exhibits a continuous, surjective homomorphism $\pi\colon \mathrm{Homeo}(X_{\beta+1,n}) \to \mathrm{Homeo}(X_{1,n})$ with coarsely bounded kernel. Thus this map is a quasi-isometry by Proposition 2.1. ☐

4 The successor ordinal case

The purpose of this section is to prove the second case of Theorem A, the case where $\alpha = \beta + 1$ is a successor ordinal.

4.1 Cayley-Abels-Rosendal graphs as Hamming graphs

In this subsection, we prove that the Cayley-Abels-Rosendal graphs given in Section 2.4 are isomorphic to Hamming graphs.

Let $\mathbf{x}$ and $\mathbf{y}$ be sequences over some alphabet A. The *Hamming distance* between $\mathbf{x}$ and $\mathbf{y}$ records the number of entries where the sequences differ. When our alphabet is $\mathbb{Z}/n\mathbb{Z}$, recall that the direct sum $\bigoplus_\omega \mathbb{Z}/n\mathbb{Z}$ may be identified with the subspace of the direct product $\prod_\omega \mathbb{Z}/n\mathbb{Z}$ comprising those sequences $(x_k)_{k\in\omega}$ for which the set $\{k : x_k \neq 0\}$ is finite. Any two sequences $\mathbf{x}$ and $\mathbf{y}$ in $\bigoplus_\omega \mathbb{Z}/n\mathbb{Z}$ are thus at finite Hamming distance.

Given $h \neq 0$ in $\mathbb{Z}/n\mathbb{Z}$ and $k \in \omega$, let $h \cdot e_k$ denote the sequence whose kth coordinate is h and all others are 0.

Definition 4.1. The *infinite Hamming graph* H_n is the Cayley graph of $\bigoplus_\omega \mathbb{Z}/n\mathbb{Z}$ with respect to the infinite generating set $S = \{h \cdot e_k : k \in \omega,\ h \in \mathbb{Z}/n\mathbb{Z} \setminus \{0\}\}$.

The induced graph metric on the vertices of H_n agrees with the Hamming distance on $\bigoplus_\omega \mathbb{Z}/n\mathbb{Z}$.

Proposition 4.2. *The graphs $\Gamma_n = \Gamma(1, n)$ and H_n are isomorphic.*

First we need a short lemma.

Lemma 4.3. *Fix a labeling of the maximal points of $X_{\alpha,n}$ by elements of $\mathbb{Z}/n\mathbb{Z}$. Given a dividing partition $\mathcal{Q}$ of $X_{\alpha,n}$, label the partition elements with elements of $\mathbb{Z}/n\mathbb{Z}$ so that the ith maximal point belongs to Q_i with indices mod n. This done, there is a bijective correspondence between dividing partitions and continuous functions $X_{\alpha,n} \to \mathbb{Z}/n\mathbb{Z}$ such that the ith maximal point is mapped to $i \in \mathbb{Z}/n\mathbb{Z}$.*

Proof. Giving $\mathbb{Z}/n\mathbb{Z}$ the discrete topology, a function $\mathcal{Q}\colon X_{\alpha,n} \to \mathbb{Z}/n\mathbb{Z}$ is continuous when each preimage $Q_i = \mathcal{Q}^{-1}(i)$ is clopen. If the ith maximal point belongs to Q_i, this is a dividing partition.

Conversely, given two dividing partitions $\mathcal{P}$ and $\mathcal{Q}$, after relabeling so that the ith maximal point belongs to P_i and Q_i, we see that the dividing partitions are equal if and only if the corresponding functions are identical. ☐

Proof of Proposition 4.2. We define a bijection Φ from vertices of Γ_n, that is dividing partitions of $X_{1,n}$, to elements of $\bigoplus_\omega \mathbb{Z}/n\mathbb{Z} = VH_n$ which we will show is a graph isomorphism.

First, fix a dividing partition $\mathcal{P}$ to serve as a basepoint, and enumerate the nonmaximal points of $X_{1,n}$ as $\{x_k\}_{k\in\omega}$. Define

$$\Phi(\mathcal{Q}) = (\mathcal{Q}(x_k) - \mathcal{P}(x_k))_{k\in\omega}.$$

Here the subtraction is done in the group $\mathbb{Z}/n\mathbb{Z}$ and we use Lemma 4.3 to identify $\mathcal{P}$ and $\mathcal{Q}$ with their corresponding continuous functions $X_{1,n} \to \mathbb{Z}/n\mathbb{Z}$. By the pigeonhole principle, if $\Phi(\mathcal{Q})$ has infinitely many nonzero terms, infinitely many of them come from some P_i, contradicting the fact that (when canonically labeled), the functions $\mathcal{Q}$ and $\mathcal{P}$ are continuous and agree on the ith maximal point. The function Φ is thus well-defined.

An inverse to Φ takes an element $(z_k)_{k\in\omega}$ of $\bigoplus_\omega \mathbb{Z}/n\mathbb{Z}$ to the function $\mathcal{Q}\colon X_{1,n} \to \mathbb{Z}/n\mathbb{Z}$ which agrees with $\mathcal{P}$ on the maximal points and assigns the nonmaximal point $x_k \in X_{1,n}$ to the partition element with index

$$\mathcal{Q}(x_k) = \mathcal{P}(x_k) + z_k$$

computed in $\mathbb{Z}/n\mathbb{Z}$. Since $\mathcal{Q}$ differs from $\mathcal{P}$ on finitely many isolated points, $\mathcal{Q}$ is a dividing partition. These operations are mutual inverses, so Φ is a bijection.

Two vertices $\Phi(\mathcal{Q})$ and $\Phi(\mathcal{Q}')$ are adjacent in H_n when their difference $\Phi(\mathcal{Q}) - \Phi(\mathcal{Q}')$ has one nonzero entry. This nonzero entry corresponds to one isolated point x that moves from $\mathcal{Q}(x)$ to $\mathcal{Q}'(x)$. The partitions $\mathcal{Q}$ and $\mathcal{Q}'$ thus differ by a 0-shift. Conversely, two partitions $\mathcal{Q}$ and $\mathcal{Q}'$ that differ by a 0-shift satisfy that $\Phi(\mathcal{Q}) - \Phi(\mathcal{Q}')$ has exactly one nonzero entry. Thus the map Φ is a graph isomorphism. ☐

The previous proposition and Corollary 3.4 have the following corollary.

Corollary 4.4. *If α is a countable successor ordinal and $n > 1$, H_n is a Cayley–Abels–Rosendal graph for* $\mathrm{Homeo}(X_{\alpha,n})$. ☐

4.2 Bi-Lipschitz Hamming graphs

The purpose of this section is to prove that all of the graphs H_n for $n > 1$ are bi-Lipschitz equivalent. The proof uses a Cantor–Schroeder–Bernstein type result for locally infinite graphs and the following "swindle".

Lemma 4.5. *The spaces*

$$\bigoplus_{\omega} \mathbb{Z}/n\mathbb{Z} \text{ and } \bigoplus_{\omega} \left(\bigoplus_{\omega} \mathbb{Z}/n\mathbb{Z} \right)$$

are isometric when $\bigoplus_{\omega} \mathbb{Z}/n\mathbb{Z}$ *is equipped with the Hamming distance and* $\bigoplus_{\omega} (\bigoplus_{\omega} \mathbb{Z}/n\mathbb{Z})$ *is equipped with the* ℓ^1 *sum of Hamming distances.*

Proof. The isometry comes from partitioning the countably infinite index set into countably infinitely many copies of itself. ☐

If we consider Cayley graphs, equipping $\mathbb{Z}/n\mathbb{Z}$ with the complete generating set and each copy of $\bigoplus_{\omega} \mathbb{Z}/n\mathbb{Z}$ in the infinite direct sum with the generating set $S = \{h \cdot e_k : k \in \mathbb{N}, h \in \mathbb{Z}/n\mathbb{Z}\}$, the isometry above yields an isomorphism of Cayley graphs. Noting that $\bigoplus_{\omega} \mathbb{Z}/n\mathbb{Z} = VH_n$, the previous lemma motivates the following definition of a direct sum of graphs.

Definition 4.6. Given a sequence of basepointed simplicial graphs $(\Gamma_i, \star_i)$ over an ordered set I, the simplicial graph $\bigoplus_{i\in I} \Gamma_i$ is the graph whose vertex set is the subset of $\prod_{i\in I} V\Gamma_i$ comprising those sequences $(v_i)_{i\in I}$ for which all but finitely many of the vertices v_i satisfy $v_i = \star_i$, where (v_i) is adjacent to (w_i) if all but one of the coordinates are equal and the remaining pair of vertices $v_j \neq w_j$ are adjacent in Γ_j.

With this language and basepoint $\mathbf{0}$, Lemma 4.5 shows that H_n and $\bigoplus_{\omega} H_n$, are isomorphic. In general, the direct sum of Hamming graphs will again be a Hamming graph. By abuse of notation we will no longer distinguish between the graph H_n and its vertex set and will always use $\mathbf{0}$ as a basepoint.

Let K_m denote the complete graph on m vertices; it is the Cayley graph of $\mathbb{Z}/m\mathbb{Z}$ with respect to the generating set $\{h \in \mathbb{Z}/m\mathbb{Z} : h \neq 0\}$.

Lemma 4.7. *For any* $n, m \geq 2$*, there is a* 4*-bi-Lipschitz bijection between the vertex sets of* H_n *and* $K_m \oplus H_n$*.*

The proof is a variant of the classic "back-and-forth" argument used to prove the Cantor–Schroeder–Bernstein theorem. Let us remark that for our application below, we really need a *bijection*. A general nonsurjective quasi-isometry, when we swindle, would leave an unbounded gap in the image of the product map.

Proof. Let $\iota\colon H_n \hookrightarrow K_m \oplus H_n$ denote the inclusion map $\iota(x) = (0, x)$; it is an isometric embedding. Let $f\colon K_m \oplus H_n \to H_n$ be the forgetful map $f(\ell, x) = x$; it is 1-Lipschitz. Observe that $f \circ \iota$ is the identity, while $\iota \circ f$ is 1-Lipschitz.

Begin by enumerating the vertices of H_n as $\{x_i\}_{i\in\mathbb{N}}$ and the vertices of $K_m \oplus H_n$ as $\{y_i\}_{i\in\mathbb{N}}$. Next we define a bijection $\psi : H_n \to K_m \oplus H_n$ by declaring $\psi(\mathbf{0}) = (0, \mathbf{0})$ and using the following "back and forth" argument. Repeat the following two steps:

1. Let $x_i \in H_n$ be the vertex with smallest index for which $\psi(x_i)$ is not yet defined. If $\iota(x_i) = (0, x_i)$ is not yet equal to $\psi(x_j)$ for $j < i$, define $\psi(x_i) = \iota(x_i)$.

   In the contrary case, there are infinitely many neighbors of $\iota(x_i)$, so we may choose one, say y, that is not yet equal to $\psi(x_j)$ for $j < i$ and define $\psi(x_i) = y$.

2. Now let $y_i \in K_m \oplus H_n$ be the vertex with smallest index that is not already equal to $\psi(x_j)$ for some $j \leq i$. If $f(y_i) \neq x_j$ for any $j \leq i$, define $\psi(f(y_i)) = y_i$.

   In the contrary case, there are infinitely many neighbors of $f(y_i)$, so we may choose one, say x, on which ψ is not yet defined and define $\psi(x) = y_i$.

Continue in this manner, alternately assigning the image of x_i's and y_i's to be within one of their image under the inclusion or forgetful map respectively. The map ψ constructed by induction is a bijection.

To see that ψ is Lipschitz, it is enough to check what happens to adjacent vertices, say x and x'. To fix notation, let $\psi(x) = (\ell, z)$ and $\psi(x') = (\ell', z')$. By construction, x and z (respectively x' and z') are either equal or adjacent in H_n. Thus we list the vertices of a path in $K_m \oplus H_n$ between $\psi(x)$ and $\psi(x')$ as

$$\psi(x) = (\ell, z), (\ell', z), (\ell', x), (\ell', x'), (\ell', z') = \psi(x').$$

This path has length at most 4, so ψ is 4-Lipschitz, as desired.

Similarly, if y and y' are adjacent in $K_m \oplus H_n$, we have a path

$$\psi^{-1}(y), f(y), f(y'), \psi^{-1}(y')$$

in H_n of length at most 3, so we see that ψ^{-1} is 3-Lipschitz. □

Theorem 4.8. *For each $n, m \geq 2$, the spaces H_n and H_m are 16-bi-Lipschitz equivalent.*

Proof. First observe that the product map

$$\begin{aligned} \Psi\colon \bigoplus_{\omega} H_n &\to \bigoplus_{\omega} (K_m \oplus H_n) \\ (z_i) &\mapsto (\psi(z_i)) \end{aligned}$$

is a 4-bi-Lipschitz bijection.

Now we have the following isomorphisms of graphs (or of spaces where each sum is equipped with the ℓ^1 metric).

$$\bigoplus_{\omega} (K_m \oplus H_n) \cong \left(\bigoplus_{\omega} K_m \right) \oplus \left(\bigoplus_{\omega} H_n \right) \cong H_m \oplus H_n.$$

So H_n is 4-bi-Lipschitz equivalent to $H_m \oplus H_n$. By symmetry, H_m is also 4-bi-Lipschitz equivalent to $H_m \oplus H_n$. Thus, H_m and H_n are 16-bi-Lipschitz equivalent. □

4.3 Assembling the pieces

We are now ready to prove the successor case of Theorem A.

Theorem 4.9. *For any countable successor ordinal α and any $n > 1$, the groups* $\mathrm{Homeo}(X_{\alpha,n})$ *and* $\mathrm{Homeo}(X_{1,2})$ *are quasi-isometric. Furthermore, they are all quasi-isometric to H_2.*

Proof. By Corollary 3.4, the group $\mathrm{Homeo}(X_{\alpha,n})$ is quasi-isometric to $\mathrm{Homeo}(X_{1,n})$. By Proposition 4.2, H_n is a Cayley–Abels–Rosendal graph for $\mathrm{Homeo}(X_{1,n})$. Therefore we have that H_n, $\mathrm{Homeo}(X_{1,n})$ and $\mathrm{Homeo}(X_{\alpha,n})$ are all quasi-isometric. By Theorem 4.8, H_n is 16-bi-Lipschitz equivalent to H_2, which is a Cayley–Abels–Rosendal graph for $\mathrm{Homeo}(X_{1,2})$. □

5 The limit ordinal case

Next we turn to the limit ordinal case of Theorem A. Since we are no longer dealing with groups that admit Cayley–Abels–Rosendal graphs, we first need to introduce some new objects.

5.1 Trees of Cayley–Abels–Rosendal graphs

The purpose of this section is to state and prove Proposition 5.3 constructing a graphical model Γ for any non-Archimedean Polish group G that is locally bounded but not generated by any coarsely bounded set. This graph Γ admits a continuous, bounded-cocompact action of G with coarsely bounded stabilizers, similar to the *coarse Cayley–Abels–Rosendal graphs* of [KS25]. Here *bounded-cocompact* means that for every closed bounded subgraph $\Lambda \subset \Gamma$, $G\Lambda/G$ is compact.

In [KS25], the authors consider vertex-transitive metric graphs with edge lengths tending to infinity. Here, our graph Γ is equipped with the graph metric, but is modeled on a tree-of-spaces construction [BH99, Chapter II.11] whose quotient graph of groups is a ray. In particular, our group action is *not* vertex transitive. Our proof of Proposition 5.3 therefore uses tools from [KS25] but does not quite follow directly from their work.

First, recall that if a group G acts on a tree T, each element $g \in G$ is either *elliptic* in that the set $\mathrm{Fix}(g) = \{x \in T : g.x = x\}$ is nonempty, or g is *hyperbolic.* If $A \subset G$, let

$$\mathrm{Fix}(A) = \{x \in T : g.x = x \text{ for all } g \in A\}.$$

Proposition 5.1. *Suppose that G is a non-Archimedean Polish group. If G is locally bounded but not generated by any coarsely bounded set, there exists a continuous action of G on a tree T with open vertex stabilizers each generated by coarsely bounded sets. The action has the property that if $A \subset G$ is coarsely bounded, then* $\mathrm{Fix}(A) \neq \varnothing$*, but G acts* nontrivially *in that* $\mathrm{Fix}(G) = \varnothing$.

Proof. Since G is locally bounded and non-Archimedean, there exists an open subgroup $H \leq G$ that is coarsely bounded in G. Because G is countably generated over the identity neighborhood H, there exists a countable set $S = \{s_k : k \in \mathbb{N}\}$ such that $H \cup S$ generates G. Fixing k, the subgroup $G_k = \langle H, s_1, \dots, s_k\rangle$ is generated by the coarsely bounded set $H \cup \{s_1, \dots, s_k\}$ so because we assume that G is not generated by a coarsely bounded set, the group G_k is a proper, open subgroup of G.

Let T be the graph with vertex set consisting of the collection of cosets gG_k and edges between vertices gG_k and gG_{k+1}. The group G acts continuously on T. Since each G_k is open, each G_k is of countable index; hence the graph T is countable. Additionally, since $G_k < G_{k+1}$, T is a tree.

If $A \subset G$ is coarsely bounded, by Rosendal's criterion, since H is open there exists a finite set F and $\ell \in \mathbb{N}$ such that $A \subset (FH)^\ell \subset \langle F, H\rangle$. Because $G = \varinjlim G_k$, each $f \in F$ is in some G_k, whence $A \subset G_K$ for some K, and hence the identity coset of G_K, thought of as a vertex of T, belongs to $\mathrm{Fix}(A)$. On the other hand, because each G_k is a proper subgroup of G, $\mathrm{Fix}(G) = \varnothing$. ☐

The quotient graph of groups $G\backslash T$ for T as in the statement of Proposition 5.1 is a ray with each vertex group G_k including into G_{k+1}. The situation is pictured in Figure 4. For $v \in T$, define $\pi(v) = k$ when the image of v in $G\backslash T$ is the kth vertex of the ray.

Definition 5.2 (Tree of Cayley–Abels–Rosendal Graphs)**.** We continue with notation as in Proposition 5.1: the non-Archimedean Polish group G has an open subgroup H coarsely bounded in G and is generated by H together with the countably infinite set $S = \{s_k : k \in \mathbb{N}\}$. For each n we have a proper, open subgroup $G_k = \langle H, s_1, \dots, s_k\rangle$.

As in [BDHL25], let Γ_k be the graph with vertex set G_k/H and an edge between gH and ghs_kH for $h \in H$ and $s_i \in S$ with i satisfying $1 \leq i \leq k$. The group G_k acts continuously and vertex-transitively on Γ_k with k orbits of edges. For each k, Γ_{k+1} contains a canonical copy of Γ_k obtained by first deleting edges labeled by s_{k+1} and then taking the connected subgraph containing the identity coset H.

Now we will construct a new graph Γ in a similar process as we did the graph T above. We first start exactly with the graph $\tilde{T}$ again. That is, $\tilde{T}$ has vertex set $G \times \mathbb{N}$ and edges connecting

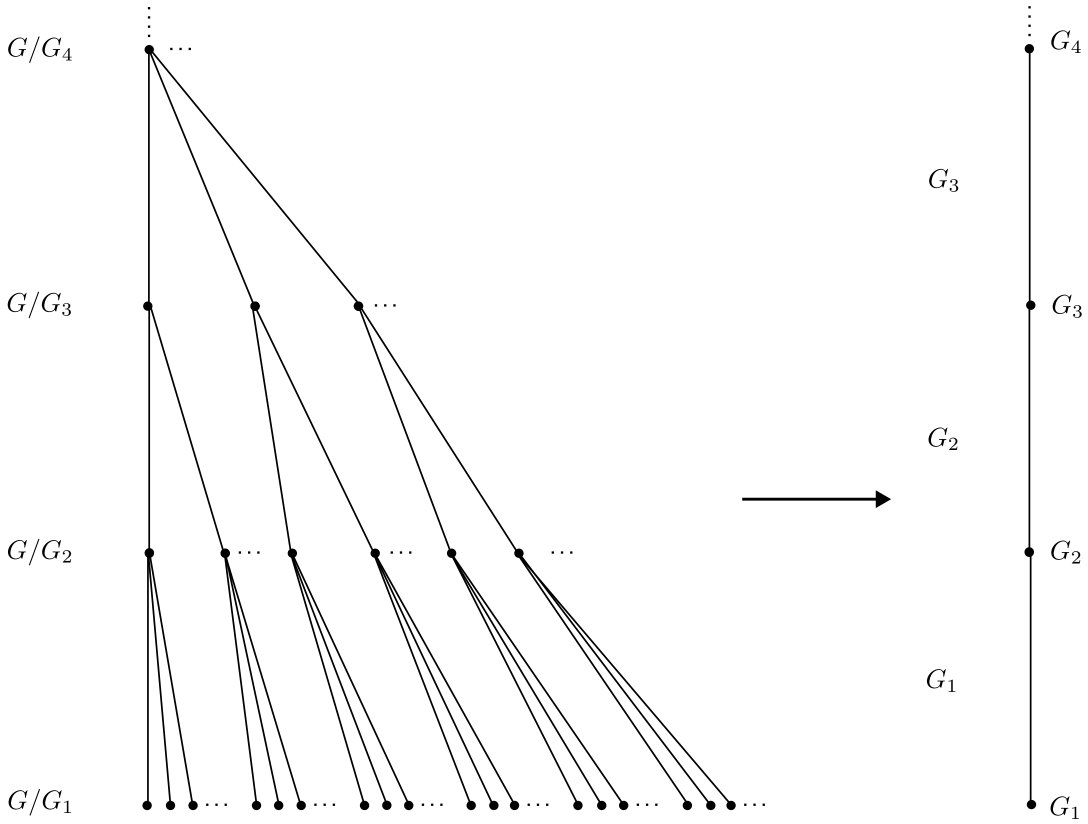

Figure 4: A schematic of the Bass-Serre T together with the quotient ray of groups G/T as in Proposition 5.1. In the tree T, all of the leftmost vertices are indexed via cosets of G_1, then the next are indexed via cosets of G_2, etc.

(g, k) to $(g, k+1)$ for each $g \in G$ and $k \in \mathbb{N}$. Next we let $\tilde{\Gamma}$ be the graph quotient of $\tilde{T}$ by the relation

$$(gh, k) \sim (g, k)$$

on the vertex set of $\tilde{T}$ where $h \in H$. We now have a bijection between the vertices of $\tilde{\Gamma}$ and the set $G/H \times \mathbb{N}$ and we will make use of this labeling of the vertices. We refer to the edges of $\tilde{\Gamma}$ as *vertical* and we refer to the second coordinate of a vertex as its *height*.

Next we form Γ from $\tilde{\Gamma}$ by adding new *horizontal edges*. Note that the cosets of H in G can be partitioned via the cosets of G_k in G for each k. Now add edges at each height k so that we have a copy of Γ_k corresponding to each coset of G_k. That is, we add an edge between vertices of the form (fH, k) and (fhs_iH, k) for $h \in H$ and $s_i \in S$ with i satisfying $1 \leq i \leq k$. This gives us a graph Γ which we call a *tree of Cayley-Abels-Rosendal graphs* for G.

We make note of a number of features of this graph Γ. Two H-cosets of the form fH and fhs_iH for $h \in H$ and $1 \leq i \leq k$ must lie in the same G_k-coset. One can picture Γ by taking the Bass-Serre tree in Figure 4 and blowing up a vertex corresponding to a coset of G_k in G by a copy of Γ_k. Indeed, for each k, let Λ_k be the subgraph of Γ of height k vertices. The connected components of Λ_k are in bijection with cosets of G_k and each connected component is isomorphic to Γ_k.

As before, Γ is a countable graph and G acts on Γ by acting on the first coordinate of a vertex by left multiplication. This action preserves heights and so has countably infinitely many orbits of vertices. Each vertex stabilizer is conjugate to the open subgroup H, so the action is continuous.

Each horizontal edge corresponds to an element $s_i \in S$. Each vertex at height k has a unique vertical edge connecting it to a vertex at height $k+1$, and when $k > 1$, a unique edge connecting to a vertex at height $k-1$. There is a graph map $\pi : \Gamma \to T$ defined on each height by collapsing each connected component of Λ_k to its corresponding G_k-coset, while not collapsing any vertical edges.

The following is adapted from the proof of [KS25, Proposition 4.3].

Proposition 5.3. *In the notation above, if $v \in \Gamma$ is a vertex, the orbit map $g \mapsto g.v$ from the action of G on Γ is a coarse equivalence between G with its canonical coarse structure and $G.v$ equipped with the subspace metric.*

Proof. We work as in the proof of [KS25, Proposition 4.3]. To be a coarse equivalence, the orbit map must *a priori* satisfy three properties: bornologous, expanding and cobounded. By [KS25, Lemma 2.10], all isometric and continuous actions have bornologous orbit maps. Because the orbit map is transitive, it is cobounded. By [KS25, Lemma 2.11], showing the orbit map is expanding is equivalent to showing that the preimage of bounded subsets in $G.v$ are coarsely bounded in G. We show this property holds to complete the proof.

Let $B \subset G.v$ be bounded, and let $A \subset G$ be its full preimage. Since B is bounded, we have a uniform bound, say M, so that $d_\Gamma(v, a.v) \leq M$ for all $a \in A$. Assuming that the height of the vertex v is N we thus have that any path from v to $a.v$ is contained in the subgraph, $\Lambda_{\leq L}$ of Γ spanned by vertices of height at most $L = N + \frac{M}{2}$. Each edge in $\Lambda_{\leq L}$ is either vertical, or a horizontal edge labeled with an element of the set $\{s_1, \ldots, s_L\} \subset S$. Thus by taking $F = \{s_1, \ldots, s_L\}$ we have that $a \in (FH)^M$. Since H is coarsely bounded in G and F is finite, the set $(FH)^M$ is coarsely bounded in G and so A is also.

□

5.2 Trees of Hamming Graphs

In this section, we define a variant TH_n of the Hamming graph H_n. These graphs have countably infinitely many "levels", where each level is a disjoint union of copies of H_n. We show that for m and $n > 1$, there is a level-preserving bijection $TH_m \to TH_n$ which is the 16-bi-Lipschitz bijection on each component from the proof of Theorem 4.8.

Definition 5.4. The graph TH_n has as vertex set

$$\omega \times \bigoplus_{\omega} VH_n.$$

A vertex $(k, (v_0, v_1, \ldots))$ is connected to all vertices of the form $(k, (w_0, v_1, \ldots))$, where v_0 and w_0 are adjacent in H_n and all other entries are equal. We also connect each vertex $(k, (v_0, v_1, \ldots))$ to the vertex $(k+1, (v_1, v_2, \ldots))$. That is, we have an edge between two vertices where one is obtained from the other by applying $+1$ in the first coordinate, and applying the left shift in the second coordinate.

The graph with vertex set

$$\omega \times \bigoplus_{\omega} VH_n$$

and only the edges connecting each vertex $(k, (v_0, v_1, \ldots))$ to the vertex $(k+1, (v_1, v_2, \ldots))$ is a tree; call it T_n.

There is a surjective graph map $TH_n \to T_n$ that sends a vertex $(k, (v_0, v_1, v_2, \ldots))$ in TH_n to the vertex $(k, (v_1, v_2 \ldots))$ in T_n obtained by applying the left shift in the second coordinate. The preimage of each vertex $(k, (v_1, v_2, \ldots))$ is a connected graph isomorphic to H_n.

The proof of the following corollary follows directly from the definitions and Theorem 4.8.

Corollary 5.5. *Let $\Phi\colon H_n \to H_m$ be the 16-bi-Lipschitz bijection from the proof of Theorem 4.8. The map $\hat{\Phi}\colon TH_n \to TH_m$ defined as $\hat{\Phi}(k, (v_1, v_2, \ldots)) = (k, (\Phi(v_1), \Phi(v_2), \ldots)$ is 16-bi-Lipschitz, preserves levels, and restricts to a 16-bi-Lipschitz bijection on each fiber of the maps $TH_n \to T_n$ and $TH_m \to T_m$.* ☐

5.3 Assembling the pieces

The purpose of this section is to prove the limit ordinal case of Theorem A. Suppose $\lambda = \lim_{k\to\infty} \alpha_k$ is a countable limit ordinal.

Fix a dividing partition $\mathcal{P}$ of $X_{\lambda,n}$. The stabilizer of $\mathcal{P}$, call it H, is open and coarsely bounded in $\mathrm{Homeo}(X_{\lambda,n})$ by Lemma 2.6. Given $k \in \mathbb{N}$, choose an α_k-shift s_k in $\mathrm{Homeo}(X_{\lambda,n})$. Because $\lim_{k\to\infty} \alpha_k = \lambda$, we have

$$\bigcup_{k=1}^{\infty} G_k = \mathrm{Homeo}(X_{\lambda,n}),$$

where $G_k = \langle H, s_1, \ldots, s_k \rangle$. However, each G_k is a proper open subgroup of $\mathrm{Homeo}(X_{\lambda,n})$.

As we showed in [BDHL25, Proposition 19], these subgroups show that $\mathrm{Homeo}(X_{\lambda,n})$ is locally bounded but not generated by any coarsely bounded set.

Proposition 5.6. *For $G = \mathrm{Homeo}(X_{\lambda,n})$, the Bass–Serre tree T, defined as in Proposition 5.1, is isomorphic to the tree T_n as defined in Definition 5.4. Let Γ be the graph defined with respect to the open subgroups G_k as in Proposition 5.1. There is a fiber-preserving, quasi-isometric graph map $\Gamma \to TH_n$ that sends vertical edges to vertical edges.*

Assuming the proposition, we complete this case of the proof of the main theorem.

Theorem 5.7. *If λ is a limit ordinal and $n > 1$, the group $\mathrm{Homeo}(X_{\lambda,n})$ is coarsely equivalent to the set of minimum-height vertices in TH_n equipped with the subspace metric. Furthermore, $\mathrm{Homeo}(X_{\lambda,n})$ is coarsely equivalent to $\mathrm{Homeo}(X_{\omega,2})$.*

Proof. The orbit map on the set of minimum-height vertices in Γ, the graph defined as in Definition 5.2, is a coarse equivalence by Proposition 5.3. By Proposition 5.6, the graphs Γ and TH_n are quasi-isometric via a quasi-isometry that preserves the fibers of the maps $\Gamma \to T$ and $TH_n \to T_n$, and hence the sets of minimum-height vertices. Thus the group $\mathrm{Homeo}(X_{\lambda,n})$ is coarsely equivalent to the set of minimum-height vertices of TH_n equipped with the subspace metric. By Corollary 5.5, since there is a level-preserving bi-Lipschitz equivalence $TH_n \to TH_m$ for n, $m > 1$, it follows that $\mathrm{Homeo}(X_{\lambda,n})$ is coarsely equivalent to $\mathrm{Homeo}(X_{\omega,2})$. ☐

Lemma 5.8. *The graph Γ_k associated to G_k as in Definition 5.2 is quasi-isometric to H_n with constants depending only on n.*

In fact, the vertices of Γ_k are in equivariant bijection with dividing partitions of $X_{\lambda,n}$ that disagree with $\mathcal{P}$ on at mostly finitely many points of rank α_k. As we see in the proof, the quasi-isometry labels and counts these points.

Proof. Fixing k, enumerate the points of rank α_k in $X_{\lambda,n}$ as $A_k = \{x_i\}_{i\in\omega}$.

Using Lemma 4.3, given a coset $gH \in G_k/H$ (i.e. a vertex of Γ_k), identify the dividing partitions $\mathcal{P}$ and $g\mathcal{P} = \mathcal{Q}$ with continuous functions $X_{\lambda,n} \to \mathbb{Z}/n\mathbb{Z}$. We define a map $\Phi = \Phi_{\mathcal{P},k} \colon \Gamma_k \to H_n$ by the rule that $\Phi(gH)$ is the sequence $(z_i)_{i\in\omega}$ of elements in $\mathbb{Z}/n\mathbb{Z}$ defined by the rule

$$z_i = \mathcal{Q}(x_i) - \mathcal{P}(x_i),$$

where as usual the difference is computed in $\mathbb{Z}/n\mathbb{Z}$. We show that each such sequence (z_i) has finitely many nonzero entries, that the map is surjective, 1-Lipschitz, and we can lift edges in H_n to paths of length at most $k(k-1)+1$, from which the lemma follows.

For the first claim, suppose to the contrary that some $(z_i) = \Phi(gH)$ has infinitely many nonzero entries. Then for infinitely many rank-α_k points x_i, the partitions $\mathcal{P}$ and $g\mathcal{P}$ disagree on x_i. By compactness of $X_{\lambda,n}$, this infinite set has a limit point x_∞, which must have rank strictly greater than α_k. By continuity, the partitions $\mathcal{P}$ and $g\mathcal{P}$ also disagree on x_∞. This contradicts the assumption that $gH \in G_k/H$, since each generator of G_k fixes all points of rank greater than α_k.

Two vertices of Γ_k are adjacent if and only if (up to rearranging) they are of the form gH and ghs_iH, where $h \in H$, s_i is our chosen α_i-shift, and $i \le k$. We show that $\Phi(ghs_iH) - \Phi(gH)$ has at most one nonzero entry, showing that the map Φ is 1-Lipschitz.

Expanding the definitions, such an entry corresponds to a rank-α_k point x_i on which $ghs_i\mathcal{P}$ and $g\mathcal{P}$ disagree, which happens if and only if $\mathcal{P}$ and $s_i\mathcal{P}$ disagree on $(gh)^{-1}(x_i)$. Therefore it suffices to show the claim for $\mathcal{P}$ and $s_i\mathcal{P}$.

If $i < k$, then the α_i-shift s_i moves no points of rank α_k between partition elements, so in this case $\Phi(s_iH) = \mathbf{0}$. If $i = k$, then by definition the α_k-shift s_k moves one point of rank α_k between partition elements, so $\Phi(s_kH)$ has exactly one nonzero entry. This proves the claim.

Since H acts transitively on pairs $(x_i, j) \in A_k \times \mathbb{Z}/n\mathbb{Z}$ such that $j \neq \mathcal{P}(x_i)$, any element $j \cdot e_i$ with exactly one nonzero entry can be obtained from considering the adjacency of $\mathcal{P}$ and $hs_kh^{-1}\mathcal{P}$ in Γ_k. Inductively, this shows that the map Φ is surjective.

Finally, choose adjacent vertices of H_n and by surjectivity say that these vertices are $\Phi(gH)$ and $\Phi(g'H)$ respectively. By definition, $g\mathcal{P}$ and $g'\mathcal{P}$ differ by at most one rank-α_k point. After applying one α_k-shift, we have that $ghs_k\mathcal{P}$ and $g'\mathcal{P}$ agree on all rank-α_k points, and thus by Lemma 3.1 differ by at most $k(k-1)$ α_k-shifts, from which the final claim follows. ☐

Proof of Proposition 5.6. The trees T and T_n both have countably infinitely many vertices at each level $k \ge 1$, each vertex has a unique edge connecting it to a vertex at the next level, and if $k > 1$, each vertex has countably infinitely many children; these trees are therefore isomorphic. Abusing notation, identify them. This done, we may identify vertices of T at level k with the cosets G/G_k.

Fix a dividing partition $\mathcal{P}$ of $X_{\lambda,n}$. For each k, choose a set of coset representatives g_{m_k} for G/G_k. This choice determines a set of partitions $g_{m_k}\mathcal{P}$.

We define a map $\Phi \colon V\Gamma \to VTH_n$ which preserves vertical adjacency and restricts to the map from Lemma 5.8 on each fiber of the map $\Gamma \to T$.

Recall that cosets of H are in equivariant bijection with dividing partitions of $X_{\lambda,n}$. We will thus work with dividing partitions. Fix k and a dividing partition $\mathcal{Q}$. We claim that there exists a unique coset $g_{m(\mathcal{Q})_k}G_k \in G/G_k$ such that $g_{m(\mathcal{Q})_k}\mathcal{P}$ and $\mathcal{Q}$ differ on at most finitely many points of rank α_k. Since each pair of representatives of distinct cosets in G/G_k differ on infinitely many points of rank α_k this follows by the pigeonhole principle.

Define

$$\Phi(\mathcal{Q}, k) = (k, (\Phi_{g_{m(\mathcal{Q})_k}\mathcal{P},k}(\mathcal{Q}), \Phi_{g_{m(\mathcal{Q})_{k+1}}\mathcal{P},k+1}(\mathcal{Q}), \ldots)).$$

It is clear from the definition that this sends the vertically adjacent pair $(\mathcal{Q}, k)$ and $(\mathcal{Q}, k+1)$ to vertices of TH_n which are adjacent by a vertical edge. Since fibers of the map $\Gamma \to T$ correspond to elements of the same G_k-coset and fibers of the map $TH_n \to T_n$ are vertices $(k, (v_1, v_2, \ldots))$ that differ only in their v_1-coordinate, the map restricts to the quasi-isometry of Lemma 5.8 on each fiber. By construction, the map Φ is surjective, 1-Lipschitz, and point preimages have diameter at most $k(k-1)$ because fibers and vertical edges are preserved and the map on fibers has these properties. Finally, by the same argument as at the end of the proof of Lemma 5.8 we see that Φ is a quasi-isometry. $\square$

6 Looking towards surfaces

One motivation for studying homeomorphism groups of Stone spaces is to better understand mapping class groups of infinite-type surfaces. For an infinite-type surface Σ, its space of ends, $E(\Sigma)$, is a second countable Stone space. Furthermore, the mapping class group, $\mathrm{MCG}(\Sigma)$, surjects onto the homeomorphism group of its end space, giving the following short exact sequence.

$$1 \longrightarrow \mathrm{PMCG}(\Sigma) \longrightarrow \mathrm{MCG}(\Sigma) \longrightarrow \mathrm{Homeo}(E(\Sigma), E_g(\Sigma)) \longrightarrow 1$$

where $E_g(\Sigma) \subset E(\Sigma)$ is the (closed) subspace corresponding to ends accumulated by genus (often referred to as non-planar ends) and $\mathrm{PMCG}(\Sigma)$ denotes the pure mapping class group of Σ. One may hope to use this short exact sequence in order to upgrade the above results to the setting of mapping class groups of infinite-type surfaces with countable end spaces. Unfortunately, as seen below, this does not work directly in most cases. That said, the techniques and results above may still offer a strategy for better understanding the coarse geometry of mapping class groups of infinite-type surfaces. In particular, given that we show that the coarse geometry of homeomorphism groups of countable Stone spaces degenerates into only three categories, if one hopes for any type of quasi-isometric rigidity among mapping class groups of infinite-type surfaces, one *must* make use of the surface topology.

We first recall a tool helpful for exhibiting the failure of a subset to be coarsely bounded. A *continuous length function* on a topological group G is a continuous map $\ell : G \to [0, \infty)$ such that $\ell(\mathrm{Id}) = 0$, $\ell(g) = \ell(g^{-1})$, and $\ell(gh) \leq \ell(g) + \ell(h)$ for all $g, h \in G$. If ℓ is a length function on G so that ℓ is unbounded on a subset $H \subset G$, then H is *not* coarsely bounded in G. In particular, given a continuous length function ℓ on G, one can obtain a continuous left-invariant pseudometric by taking $d(g, h) = \ell(g^{-1}h)$ for all $g, h \in G$.

Now we can utilize a length function defined in [MR23, Section 2] to immediately show that the quotient map above is not a coarse equivalence for a large family of surfaces. We say that a subsurface $K \subset \Sigma$ is *nondisplaceable* if K and $f(K)$ intersect essentially for all $f \in \mathrm{Homeo}(\Sigma)$.

Lemma 6.1. *If Σ has a finite-type nondisplaceable subsurface, then $\mathrm{PMCG}(\Sigma)$ is not coarsely bounded in $\mathrm{MCG}(\Sigma)$. In particular, the quotient map $\mathrm{MCG}(\Sigma) \to \mathrm{Homeo}(E(\Sigma), E_g(\Sigma))$ is not a coarse equivalence.*

Proof. In [MR23, Section 2], given a finite-type nondisplaceable subsurface $K \subset \Sigma$, the authors construct a continuous length function ℓ on $\mathrm{MCG}(\Sigma)$ that is unbounded on the subgroup $\langle f \rangle$ for any mapping class $f \in \mathrm{MCG}(\Sigma)$ that preserves K and restricts to a pseudo-Anosov mapping class on K. Now, picking a finite-type subsurface K in Σ of sufficient complexity, there exists a pseudo-Anosov mapping class $g_0 \in \mathrm{PMCG}(K)$. By extending g_0 via the identity we obtain a pure mapping class $g \in \mathrm{PMCG}(\Sigma)$ for which ℓ is unbounded on $\langle g \rangle$. $\square$

Next we use length functions to check that the puncture-forgetting "Cantor-Bendixson derivative" does not induce a coarse equivalence on mapping class groups of surfaces, as opposed to the case of countable Stone spaces.

Lemma 6.2. *Let Σ be a surface with at least one isolated planar end and a finite-type nondisplaceable subsurface and let Σ' be the surface Σ obtained by forgetting all isolated planar ends. The kernel of the forgetful map $\mathcal{F} : \mathrm{MCG}(\Sigma) \to \mathrm{MCG}(\Sigma')$ is not coarsely bounded and hence $\mathcal{F}$ does not induce a coarse equivalence.*

Proof. We repeat the same proof as above, now using a finite-type nondisplaceable subsurface $K \subset \Sigma$ that contains at least one isolated planar end, e. Any map obtained by point pushing e about a filling curve of K is a pseudo-Anosov mapping class in $\mathrm{PMCG}(K)$ [Kra81] and hence can be extended via the identity to a map in $\ker(\mathcal{F})$ on which the Mann-Rafi length function is unbounded. $\square$

Both of these lemmas require the existence of a non-displaceable subsurface. In [BNQR26], the authors introduce a notion of "complexity" for stable infinite-type surfaces with CB-generated mapping class groups. Roughly speaking, the natural number $\zeta(\Sigma)$ measures the minimal complexity of a finite-type subsurface used to define a CB-generating set. We refer the reader to [BNQR26, Section 2.5] for the formal definition. Notably, when Σ has either zero or infinite genus, if $\zeta(\Sigma) = 1$, then $\mathrm{MCG}(\Sigma)$ is itself CB, if $\zeta(\Sigma) = 2$, then Σ does not contain a finite-type nondisplaceable subsurace, and if $\zeta(\Sigma) \geq 3$, then Σ has a finite-type nondisplaceable subsurface. Furthermore, if $\zeta(\Sigma) = 1$ then either $E_g(\Sigma) = E(\Sigma)$ or $E_g(\Sigma) = \emptyset$ and $E(\Sigma)$ is self-similar. Thus $\mathrm{Homeo}(E(\Sigma))$ is also itself CB. This leaves one case in which we have not covered in the above lemmas, begging the question asked in the introduction, repeated below.

Question 1. Let Σ be a stable surface with $\zeta(\Sigma) = 2$ and CB-generated, but not CB, mapping class group. Is $\mathrm{PMCG}(\Sigma)$ coarsely bounded in $\mathrm{MCG}(\Sigma)$? More specifically, let $\Sigma_{1,2}$ be the genus zero surface with end space $X_{1,2}$, is $\mathrm{PMCG}(\Sigma_{1,2})$ coarsely bounded inside of $\mathrm{MCG}(\Sigma_{1,2})$?

Notably, this is exactly the case of "translatable surfaces" studied in [SC24] and "avenue surfaces" considered in [HQR22]. In [SC24] the author provides a Cayley-Abels-Rosendal graph for these mapping class groups. While we would be surprised if the answer to the above question were "yes," the current tools used in [MR23] to certify that a mapping class group is not CB cannot be used to certify that these pure mapping class groups are not CB. In particular, in [MR23, Section 7] the authors define an unbounded length function on $\mathrm{MCG}(\Sigma)$ that is actually bounded on $\mathrm{PMCG}(\Sigma)$.

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